STOCHASTIC PROCESSES

UNIT - I

Stochastic Processes, Markov chain with stationary, transition probabilities, properties of transition functions.

UNIT - II

Classification of states, stationary distribution of a Markov chain existence and uniqueness, convergence to the stationary distribution.

UNIT - III

Markov pure jump processes, Poisson process, Birth and deth process.

UNIT - IV

Second order process, mean and covariance function, Gaussian and Weiener process.

UNIT - V

Stochastic differential equations, estimation theory and spectral distribution.

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Stochastic Process :- Stochastic process means a process whose results depend on some chance elements.

A stochastic or random process can be defined as a collection of random variables that is indexed by some mathematical set, meaning that each random variable of the stochastic process is uniquely associated with an element in the set. The set used to index the random variables is called the 'idex set'. Historically, the idex set was some subset of the real line, such as the natural no., giving the index set the interpretation of time.

Each random variable in the collection takes values from the same mathematical space known as the 'state space'. The state space can be, for example, the integers, the real line or n- diamensional Euclidean space.

Examples :-

1. Size of a queue in railway booking counter.
2. Growth of a bacterial population in a laboratory culture.
3. Bernoulli process
4. Poisson process
5. Change the share prize in stock market.

Definition :-

A stochastic process may be defined as a family of random variables $\{X_t, t \in T\}$ where T is an index set. For each t, let X_t be the sample space of X_t, $X_t = \{x_t\}$, the set of all possible values x_t of X_t, which may be finite or infinite. The random variables X_t, X_{t+r} $(r>0)$ may be dependent or independent. The index set T may be finite or infinite. Often t is considered as a time parameter.

State Space :- The values assumed by a random variable $X(t)$ are called "states" and the collection of all possible values forms the "State space" (S) of the process.

If $X(t) = i$, then we say the process is in state i.

★ **Discrete state process** - The state space is finite or countable for example the non-negative integers $\{0, 1, 2, ---\}$

★ **Continuous - state process** - The state space contains finite or infinite intervals of the real number line.

Classification of stochastic process :-

A stochastic process can be classified in different ways, for example, by its state space, its index set, or the dependence among the random variable.

★ **Discrete / Continuous time :-**

If the index set of a stochastic process has a finite or countable number of elements, such as a finite set of numbers, the set of integers, or the natural numbers, then the stochastic process is said to be in discrete time. This type of stochastic process is referred to as 'Discrete - time stochastic process'.

→ If the index set of stochastic process is some interval of the real line, then time is said to be continuous and stochastic process is referred as continuous - time stochastic process.

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★ Discrete / Continuous state space :-

If the state space is the integers or natural numbers, then the stochastic process is called a ~~discrete~~ discrete or integer - valued stochastic process.

↗ If the state space is ~~reffered to as a~~ the real line, then the stochastic process is referred to as a real - valued stochastic process or a process with continuous state space.

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Classification : Depending on the continuous or discrete nature of the state space S and and parameter set T, a random process can be classified into four types :-

1. If both T and S are discrete, the random process / stochastic process is called discrete random sequence.

For example :-

(i) If X_n represents the outcome of n^{th} toss of fair dice, then $\{X_n, n \geq 1\}$ is discrete random sequence, since -

$T = \{1, 2, 3, ---\}$ and $S = \{1, 2, 3, 4, 5, 6\}$

2. If T is discrete and S is continuous, the random process is called a continuous random sequence.

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For example :-

(i) If X_n, represents the temprature at the end of the n^{th} hour of a day, then $\{X_n, 1 \leq n \leq 24\}$ is a continuous random sequence, since temperature can take any value in an interval and hence continuous.

3. If T is continuous and S is discrete, the random process is called a disrete random process.

For example :- if $X(t)$ represents the number of telephone calls received in the interval $(0, t)$ then $\{X(t)\}$ is a discrete random process, since $S = \{0, 1, 2, 3, ----\}$

The word 'discrete' or 'continuous' is used to refer to the nature of S and the word 'sequence' or 'process' is used to refer to the nature of T.

4. If both T and S are continuous, the random process is called a continuous random process.

For example :-

(i) If $X(t)$ represents the maximum temperature at a place in the interval $(0, t)$, $\{X(t)\}$ is a continuous random process.

The Poisson Process :-

The poisson distribution may also be obtained independently (i.e. without considering it as a limiting form of the binomial distribution) as follows :-

Let X_t be the number of telephone calls received in time interval 't' on a telephone switch board. Consider the following experimental conditions :-

1.) The probability of getting a call in small time interval $(t, t+dt)$ is λdt, where λ is a positive constant and 'dt' denotes a small increment in time 't'.

2) The probability of getting more than one call in this time interval is very small, i.e. is of the order of $(dt)^2$ i.e. $O[(dt)^2]$ such that

$$\lim_{dt \to 0} \frac{O[(dt)^2]}{dt} = 0$$

3.) The probability of any particular call in the time interval $(t, t+dt)$ is independent of the actual time t, and also of all previous calls.

Under these conditions it can be shown that the probability of getting x calls in time 't', say $P_x(t)$ is given by

$$P_x(t) = \frac{e^{-\lambda t} (\lambda t)^x}{x!} \quad ; \quad x = 0, 1, 2, \ldots \infty$$

which is a poisson distribution with parameter λt.

Proof Let $P_x(t) = P$[of getting x calls in a time interval of length 't']

Also P[of ~~more~~ at least one call during (t, dt)]

$$= \lambda dt + O[(dt)^2]$$

and P[of more than one call during (t, dt)]

$$= O[(dt)^2]$$

The event of getting exactly x calls in time $t + dt$ can materialise in the following two mutually exclusive ways :-

(i) x calls in $(0, t)$, and ~~more~~ none during during $(t, t+dt)$ and the probability of this event is :

$$P_x(t) \cdot [1 - \{\lambda dt + O(dt)^2\}]$$

(ii) exactly $(x-1)$ calls during $(0, t)$ and one call in $(t, t+dt)$ and the probability of this event is

$$P_{x-1}(t) \, (\lambda dt)$$

Hence by the addition theorem of probability, we get

$$P_x(t+dt) = P_x(t)\{1 - \lambda dt - O(dt)^2\} + P_{x-1}(t)\cdot \lambda dt$$
$$= P_x(t) - P_x(t)\lambda dt - P_x(t)O(dt)^2 + P_{x-1}(t)\cdot \lambda dt \quad \leftarrow (1)$$

$$\Rightarrow P_x(t+dt) - P_x(t) = dt\left[-\lambda P_x(t) + \lambda P_{x-1}(t) - P_x(t)\cdot \frac{O(dt)^2}{dt}\right]$$

$$\Rightarrow \frac{P_x(t+dt) - P_x(t)}{dt} = -\lambda P_x(t) + \lambda P_{x-1}(t) - P_x(t)\cdot \frac{O(dt)^2}{dt}$$

Proceeding to the limit as $dt \to 0$, we get

$$\lim_{dt\to 0} \frac{P_x(t+dt) - P_x(t)}{dt} = -\lambda P_x(t) + \lambda P_{x-1}(t) - P_x(t)\cdot 0$$

[From experimental condition]

$$\Rightarrow \lim_{dt\to 0} \frac{P_x(t+dt) - P_x(t)}{dt} = -\lambda P_x(t) + \lambda P_{x-1}(t)$$

$$\Rightarrow P_x'(t) = -\lambda P_x(t) + \lambda P_{x-1}(t), \quad x \geq 1 \quad \leftarrow (2)$$

Where (') denotes differentiation w. r. to 't'

For $x = 0$, $P_{x-1}(t) = P_{-1}(t)$
$= P[(-1) \text{ calls in time } 't'] \equiv 0$

Hence, from (1) we get

$$P_0(t+dt) = P_0(t) - P_0(t)\cdot \lambda dt - P_0(t)\,O(dt)^2 + P_{-1}(t)\cdot \lambda dt$$

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$$P_0(t+dt) = P_0(t)[1-\lambda dt] - P_0(t).\,O(dt)^2 + O$$

which on taking the limit $dt \to 0$, gives

$$P_0'(t) = -\lambda P_0(t)$$

$$\Rightarrow \frac{P_0'(t)}{P_0(t)} = -\lambda$$

Integrating w. r. to 't',

$$\log P_0(t) = -\lambda t + C$$

Where C is an arbitary constant to be determined from the condition $P_0(0) = 1$

Hence $C = \log 1 = 0 \Rightarrow \boxed{C=0}$

$$\therefore \log P_0(t) = -\lambda t$$

$$\Rightarrow P_0(t) = e^{-\lambda t}$$

Substituting this value of $P_0(t)$ in eq^n ②, we get, with $x=1$

$$P_1'(t) = -\lambda P_1(t) + \lambda e^{-\lambda t}$$

$$\Rightarrow P_1'(t) + \lambda P_1(t) = \lambda e^{-\lambda t}$$

This is an ~~or~~ ordinary linear differential equation whose integrating factor is $e^{\lambda t}$.

[Ordinary linear Differential Equation

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$$\frac{dy}{dx} + Py = Q$$

where P & Q are constant or function of x only

then it's solution is

$$y \cdot (I.F) = \int Q \cdot I.F. + C$$

where $I.F. = e^{\int P dx}$

For $P_1'(t) + \lambda P_1(t) = \lambda e^{-\lambda t}$

$$I.F. = e^{\int \lambda dt} = e^{\lambda t}$$

& its solution is

$$P_1(t) \cdot e^{\lambda t} = \int \lambda e^{-\lambda t} \cdot e^{\lambda t} dt + C_1$$

$$\Rightarrow P_1(t) \cdot e^{\lambda t} = \lambda \int dt + C_1$$

$$\Rightarrow P_1(t) \cdot e^{\lambda t} = \lambda t + C_1$$

where C_1 is an arbitary constant to be determined from $P_1(0) = 0$ which gives $\boxed{C_1 = 0}$

$$\therefore P_1(t) = \lambda t \cdot e^{-\lambda t}$$

Again substituting this in (2) with $x = 2$,

$$P_2'(t) + \lambda P_2(t) = \lambda e^{-\lambda t} \cdot \lambda t$$

Integrating factor of this equation is $e^{\lambda t}$ and its solution is :

$$P_2(t)\, e^{\lambda t} = \lambda^2 \int t\, e^{-\lambda t} e^{\lambda t}\, dt + C_2$$

$$= \frac{\lambda^2 t^2}{2} + C_2$$

Where C_2 is an arbitary constant to be determined from $P_2(0) = 0$, which gives $\boxed{C_2 = 0}$

Hence $P_2(t) = e^{-\lambda t} \cdot \frac{(\lambda t)^2}{2}$

Proceeding similarly step by step, we shall get

$$P_x(t) = \frac{e^{-\lambda t} (\lambda t)^x}{x!} \quad ; \quad x = 0, 1, 2, \ldots \infty$$

which is the p.m.f. of Poisson distribution with parameter λt.